Partial derivatives

Calc III $z = f(x,y)$.

The graph is a surface. You can stand on the surface & walk in the positive x-direction or the positive y-direction.

Definition:
The partial derivative of $f(x,y)$ with respect to x at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

Def:
The partial derivative of $f(x,y)$ with the respect to y at the point (x_0,y_0)

is $$\left.\frac{\partial f}{\partial y}\right| = \lim \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

$$\Big|_{(x_0,y_0)} \qquad {}_{h\to 0} \qquad h$$

$$= \frac{d}{dy} f(x_0,y)$$

Ex) Find the values of $\frac{df}{dx}$ & $\frac{df}{dy}$ at the point $(4,-5)$ if $f(x,y)=x^2+3xy+y-1$

To find $\frac{df}{dx}$, we treat y as a constant & differentiate with respect to x.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2+3xy+y-1\right)$$

$$= 2x+3y\Big|_{(4,-5)} = -7$$

$$\left.\frac{\partial f}{\partial y}\right|_{(4,-5)} = 3x+1\Big|_{(4,-5)} = 13$$

Ex| Find $\frac{df}{dy}$ as a function

if $f(x,y) = y \sin xy$

$f_y = \sin xy + xy \cos(xy)$

Ex| $f(x,y) = \frac{2x}{y+\cos x}$ $\begin{matrix} f \\ g \end{matrix}$ $g' = 0 - \sin x$

$f_x = \frac{df}{dx}$ $f_y = \frac{df}{dy}$

$$f_x = \frac{2(y+\cos x) - (2x)(-\sin x)}{(y+\cos x)^2}$$

$$= \frac{2y+2\cos x+2x\sin x}{(y+\cos x)^2}$$

$$f = 2x(y+\cos x)^{-1}$$

$$f_y = -2x\,(y+\cos x)^{-2}\cdot(1)$$

$$= \frac{-2x}{(y+\cos x)^2}$$

*) Find $\frac{dz}{dx}$,

$$yz - \ln z = x+y$$

$$\frac{dz}{dx}\left(yz - \ln z = x+y\right)$$

$$\frac{dz}{dx}(yz) - \frac{dz}{dx}(\ln z) = (x+y)\frac{dz}{dx}$$

$$y\frac{d}{dx} - \frac{1}{z}\frac{d}{dx} = 1 + 0$$

We differentiate both sides of the eqn. with respect to x, holding y constant & treating z as a differentiate function of x.

$$y\frac{dz}{dx} - \frac{1}{z}\frac{dz}{dx} = 1+0$$

$$\frac{dz}{dx}\left(y - \frac{1}{z}\right) = 1$$

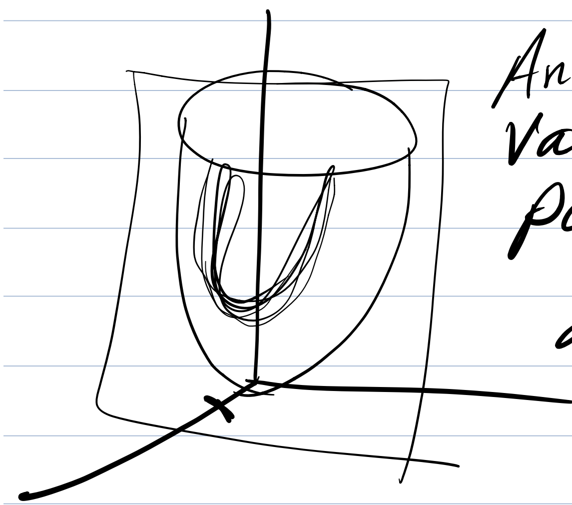
$$\frac{dz}{dx} = \frac{(1)\,z}{\left(y - \frac{1}{\cancel{z}}\right)(z)}$$

$$\frac{dz}{dx} = \frac{z}{3y-1}$$

Ex) The plane $x=1$ intersects the paraboloid

$z = x^2+y^2$ in a parabola

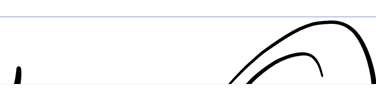
Find the slope of the tangent of the parabola at $(1,2,5)$



Ans: The slope is the value of the partial derivative

$\frac{dz}{dy}$ at $(1,2)$:

$$\left.\frac{dz}{dy}\right|_{(1,2)} = \left.\frac{d}{dy}\left(x^2+y\right)\right|_{(1,2)}$$



$$= 2y \Big|_{(1,2)} = 4 \text{ slope.}$$

Second order of partial derivatives

Ex/ $f(x,y) = x\cos y + ye^x$

Find $\dfrac{d^2f}{dx^2}$, $\dfrac{d^2f}{dydx}$, $\dfrac{d^2f}{dy^2}$, $\dfrac{d^2f}{dxy}$

— f_{xx}, f_{yx}, f_{yy}, f_{xy} —

f_{xx}: $f_x = \cos y + ye^x$

$f_{xx} = ye^x$

f_{xy}: $f_{xy} = -\sin y + e^x$

f_{yx} : $f_y = -x\sin y + e^x$

$f_{yx} = -\sin y + e^x$

f_{yy} : $f_{yy} = -x\cos y$

Ex| $w = xy + \dfrac{e^y}{y^2+1}$

find

$w_{xy} =$

$w_x = y$

$w_{xy} = 1$

$w_{yx} = ?$

Ex 1 $f(x,y,z) = 1 - 2xy^2z + x^2y$

$f_{yxyz} =$

$f_y = -4xyz + x^2$

$f_{yx} = -4yz + 2x$

$f_{yxy} = -4z$

$f_{yxyz} = -4$

Thm: The increment theorem for functions of two variables:

Suppose that the first partial derivatives of $f(x,y)$ are defined throughout an open region R containing the point (x_0, y_0) & f_x & f_y are continuous at (x_0, y_0).

Then the change

$$\nabla z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$$

in the value of f that results from moving from (x_0, y_0) to another point

$(x_0+\Delta x, y_0+\Delta y)$ in R satisfies an equation of the form

$$\nabla z = f_x(x_0, y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of $\varepsilon_1, \varepsilon_2 \to 0$

as the both $\Delta x, \Delta y \to 0$.

<u>Corollary</u>: If the partial derivatives f_x & f_y of a function $f(x,y)$ are continuous througout an open region R, then f is differentiable at every point of R.

Thm: Differentiability implies continuity:

If a function $f(x,y)$ is differentiable at (x_0, y_0) then f is continuous at (x_0, y_0)

Ex) $f(x,y,z) = \ln(x+2y+2z)$

find f_x, f_{yy}, & f_z

$$f_x = \frac{1}{x+2y+2z} \cdot (1) \qquad f_{yy} = \frac{-4}{(x+2y+2z)^2}$$

$$f_y = \frac{1}{x+2y+2z}(2)$$

$$f_z = \frac{2}{x+2y+2z}$$

Ex) $f(x,y,z) = yz\ln(xy)$

Find f_x, f_y, & f_z.

$$f_x = \frac{yz}{x\cancel{y}} \cdot \cancel{y} = \frac{yz}{x}$$

$$f_y = z\ln(xy) + \frac{\cancel{y}z}{\cancel{x}\cancel{y}} \cdot \cancel{x}$$

$$= z\ln(xy) + z$$

$$f_z = y\ln(xy)$$

Ex) $w = \ln(2x+3y)$

$$w_{xy} = -\frac{6}{(2x+3y)^2}$$

Ex) $f(x,y,z) = e^{-xyz}$

$$f_x = -yz\, e^{-xyz}$$

$$f_y = -xz\, e^{-xyz}$$

$$f_z = -xy\, e^{-xyz}$$

Ex) $w = \dfrac{x-y}{x^2+y}$

Calculate 2nd order partial derivatives.

$$\frac{\partial w}{\partial x} = \frac{(x^2+y) - (x-y)(2x)}{(x^2+y)^2} = \frac{-x^2+2xy+y}{(x^2+y)^2}$$

$$\frac{\partial w}{\partial y} = \frac{(-1)(x^2+y) - (x-y)(1)}{(x^2+y)^2} = \frac{-x^2-x}{(x^2+y)^2}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) =$$

$$\frac{(-2x+2y)(x^2+y)^2 - (-x^2+2xy+y)\,2(x^2+y)(2x)}{(x^2+y)^4}$$

$$\frac{(-2x+2y)(x^2+y) - (-x^2+2xy+y)\,4x}{(x^2+y)^3}$$

$$\frac{(-2x+2y)(x^2+y)+4x^3-8x^2y-4xy}{(x^2+y)^3} = \frac{2(x^3-3x^2y-3xy+y^2)}{(x^2+y)^3}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = \frac{0(x^2-y)^2+(x^2+x)\,2(x^2+y)(1)}{(x^2+y)^4}$$

$$= \frac{(x^2+x)\,2\cancel{(x^2+y)}}{(x^2+y)^{\cancel{4}}} = \frac{2(x^2+x)}{(x^2+y)^3}$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{-x^2-x}{(x^2+y)^2}\right) = \frac{(-2x-1)(x^2+y)^{\cancel{2}}+(x^2+x)(2\cancel{(x^2+y)})(2x)}{(x^2+y)^{\cancel{4}\,3}}$$

$$= \frac{(-2x-1)(x^2+y)+4x^2+4x}{(x^2+y)^3}$$